MULTILINEAR OPERATORS ON SPACES OF CONTINUOUS FUNCTIONS

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Dedicated to Professor Julian Musielak, on the occasion of his 70th birthday

1. Introduction

The classical Riesz's representation theorem permits to represent the continuous linear forms on C(K) as integrals with respect to Radon measures. The deep insight given by Grothendieck ([7]) into the space C(K) in terms of operators defined on it, is based on suitable representation theorems of such operators as integrals with respect to some vector measures. The natural question of the study of operators on vector-valued continuous function spaces has been also intensively treated (see, f.i., [3] and [6], and the bibliography included, or the forthcoming book [2]). The basic approach to all such representation theorems consists of extending the operador T from the space of cotinuous functions to some wider space, containing the Baire or Borel simple functions, which allows us to define the representing measure of T. The simplest way to do that is just to consider the double transpose T^{**} of T.

In this paper we are concerned with the representation of polynomials or, more generally, continuous multilinear mappings on spaces of continuous functions. The case of bilinear mappings has been considered in some extend (see [9, 11] and, especially, [12]), and it is essentially simpler than the case $n > 2$. The general case has been studied by I. Dobrakov, as part of his long series of papers on integration in Banach spaces and representation of operators (*cfr.* [5] and the references included). His point of departure is a deep result of Pelczynski [10, Theorem 2] which essentially assures the existence, under suitable hypothesis, of an extension of the continuous multilinear map $T : C(K_1) \times \cdots \times C(K_n) \to F$ to a continuous multilinear map $\overline{T} : \mathcal{B}^\Omega(K_1) \times \cdots \times \mathcal{B}^\Omega(K_n) \to F$, where $\mathcal{B}^\Omega(K)$ denotes the Banach space of all the bounded *Baire* functions on K. The limitation to the Baire functions is basic in the proof of the theorem, which uses transfinite induction, starting from $C(K_i)$ to arrive to $\mathcal{B}^\Omega(K_i)$. One of our main results is a generalization of Pelczynski's theorem, proving that there exists a *unique* continuous multilinear extension to $C(K_1)^{**} \times \cdots \times C(K_d)^{**}$, which is separately weak-* continuous. As a consequence, we obtain representation theorems for Banach valued multilinear mappings on this type of spaces in terms of polymeasures, extending and generalizing previous results ([12, 5]). Some consequences and applications are also obtained.

For the general background and standard notations on Banach spaces we refer to [8]. In particular, if E is a Banach space, B_E stands for its closed unit ball, and E^* for its topological dual. $\mathcal{L}^d(E_1, \dots E_d; F)$ denotes the Banach space of all the continous d-linear maps from $E_1 \times \ldots \times E_d$ to F. We shall omit the d when $d = 1$. Throughout the paper, K will stand for a compact Hausdorff topological space and

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$C(K)$ for the Banach space of all the (scalar) continuous functions on K, endowed with the supremum norm.

2. The results

Our first result is a general extension theorem for multilinear operators.

Theorem 1. *Let $E_1, \ldots, E_d, F$ be Banach spaces such that for all $i \neq j$, every linear operator from E_i into E_j^* is weakly compact. Then, if T is a continuous d-linear mapping from $E_1 \times \ldots \times E_d$ into F, there is a unique bounded multilineal operator $T^{**} \in \mathcal{L}^d(E_1^{**}, \ldots, E_d^{**}; F^{**})$ which extends T and is $\omega^* - \omega^*$-separately continuous. Moreover, $\|T^{**}\| = \|T\|$.*

Proof. Uniqueness follows from the ω^*-density of each E_k in E_k^{**}.

As for the existence, let us consider first the case $F = \mathbb{K}$.

If $d = 1$ the result is obvious. Let us suppose it true for $d = k-1$. For $1 \leq i \leq k-1$ let x_i be a fixed point of E_i and define $T_{x_1,\ldots,x_{k-1}}(x) = T(x_1, \ldots, x_{k-1}, x) \, \forall x \in E_k$. Then $T^{**}_{x_1,\ldots,x_{k-1}}$ is the only ω^*-continuous extension of $T_{x_1,\ldots,x_{k-1}}$ to E_k^{**}, and $\|T^{**}_{x_1,\ldots,x_{k-1}}\| = \|T_{x_1,\ldots,x_{k-1}}\|$.

Let us now define for each $x^{**} \in E_k^{**}$:

$$T_{x^{**}}(x_1, \ldots, x_{k-1}) = T^{**}_{x_1,\ldots,x_{k-1}}(x^{**})$$

Obviously $T_{x^{**}} \in \mathcal{L}^{k-1}(E_1, \ldots, E_{k-1}; \mathbb{K})$. Then, according to our induction hypothesis there exists $T^{**}_{x^{**}} \in \mathcal{L}^{k-1}(E_1^{**}, \ldots, E_{k-1}^{**}; \mathbb{K})$ that extends $T_{x^{**}}$, is ω^*-separately continuous and verifies $\|T^{**}_{x^{**}}\| = \|T_{x^{**}}\|$. Let us set $T^{**}(x_1^{**}, \ldots, x_{k-1}^{**}, x^{**}) = T^{**}_{x^{**}}(x_1^{**}, \ldots, x_{k-1}^{**})$. Then we have

$$\|T^{**}\| = \sup_{\|x_k^{**}\| \leq 1} \|T^{**}_{x_k^{**}}\| = \sup_{\|x_k^{**}\| \leq 1} \|T_{x_k^{**}}\| =$$
$$= \sup_{\substack{\|x_i\| \leq 1 \\ 1 \leq i \leq k-1}} \|T^{**}_{x_1,\ldots,x_{k-1}}\| = \sup_{\substack{\|x_i\| \leq 1 \\ 1 \leq i \leq k-1}} \|T_{x_1,\ldots,x_{k-1}}\| = \|T\|$$

For $1 \leq i \leq k-1$ let $x_i \in E_i$ be arbitrarily fixed points. The operator

$$E_1 \ni x_1 \mapsto \theta_1(x_1) = T_{x_1,\ldots,x_{k-1}} \in E_k^*$$

is, by hypothesis, weakly compact, and so its bitranspose θ_1^{**} is E_k^*-valued and $\omega^* - \omega-$continuous. So, for $x_1^{**} \in E_1^{**}$, let $(x_1^i)_{i \in I} \subset E_1$ be a net such that $x_1^i \xrightarrow{\omega^*} x_1^{**}$; then we have $\theta_1^{**}(x_1^{**}) = \omega - \lim \theta_1(x_1^i) = \omega - \lim T_{x_1^i,\ldots,x_{k-1}}$, i.e. $\forall x_k^{**} \in E_k^{**}$, $\lim \langle T_{x_1^i,\ldots,x_{k-1}}, x_k^{**} \rangle = \langle \theta_1^{**}(x_1^{**}), x_k^{**} \rangle$. But, according to the definition, $\langle T_{x_1^i,\ldots,x_{k-1}}, x_k^{**} \rangle = T^{**}_{x_1^i,\ldots,x_{k-1}}(x_k^{**}) = T^{**}(x_1^i, \ldots, x_{k-1}, x_k^{**}) \to T^{**}(x_1^{**}, x_2, \ldots, x_{k-1}, x_k^{**})$, the last limit being precisely our induction hypothesis. So we get

$$\theta_1^{**}(x_1^{**}) = T^{**}(x_1^{**}, x_2, \ldots, x_{k-1}, \cdot)|_{E_k}$$

and

$$\langle \theta_1^{**}(x_1^{**}), x_k^{**} \rangle = T^{**}(x_1^{**}, x_2, \ldots, x_{k-1}, x_k^{**}).$$

If we now consider

$$E_2 \ni x_2 \mapsto \theta_2(x_2) = T^{**}(x_1^{**}, x_2, \ldots, x_{k-1}, \cdot)|_{E_k} \in E_k^*$$

we get as before $\theta_2^{**}(x_2^{**}) = T^{**}(x_1^{**}, x_2^{**}, x_3, \ldots, x_{k-1}, \cdot)|_{E_k}$ and

$$\langle \theta_2^{**}(x_2^{**}), x_k^{**} \rangle = T^{**}(x_1^{**}, x_2^{**}, x_3, \ldots, x_{k-1}, x_k^{**}), \quad \forall x_k^{**} \in E_k^{**}$$

(i.e. $[T^{**}(x_1^{**}, x_2^{**}, x_3, \ldots, x_{k-1}, \cdot)|_{E_k}]^{**} = T^{**}(x_1^{**}, x_2^{**}, x_3, \ldots, x_{k-1}, \cdot)$).

Repeating the process $(k-1)$ times, we get

$$\langle (T^{**}(x_1^{**}, x_2^{**}, \ldots, x_{k-1}^{**}, \cdot)\big|_{E_k}), x_k^{**}\rangle = T^{**}(x_1^{**}, x_2^{**}, \ldots, x_{k-1}^{**}, x_k^{**}),$$

for every $x_i^{**} \in E_i^{**}$, $1 \leq i \leq k$.

Therefore, if $(x_k^i)_{i\in I} \subset E_k$ is such that $x_k^i \xrightarrow{\omega^*} x_k^{**}$, then

$$\lim_i \langle (T^{**}(x_1^{**}, x_2^{**}, \ldots, x_{k-1}^{**}, \cdot)\big|_{E_k}), x_k^i\rangle = \langle T^{**}(x_1^{**}, x_2^{**}, \ldots, x_{k-1}^{**}, \cdot)\big|_{E_k}, x_k^{**}\rangle.$$

Since

$$\langle (T^{**}(x_1^{**}, x_2^{**}, \ldots, x_{k-1}^{**}, \cdot)\big|_{E_k}), x_k^i\rangle = T^{**}(x_1^{**}, x_2^{**}, \ldots, x_{k-1}^{**}, x_k^i)$$

and

$$\langle (T^{**}(x_1^{**}, x_2^{**}, \ldots, x_{k-1}^{**}, \cdot)\big|_{E_k}, x_k^{**}\rangle = T^{**}(x_1^{**}, x_2^{**}, \ldots, x_{k-1}^{**}, x_k^{**}),$$

we obtain

$$T^{**}(x_1^{**}, x_2^{**}, \ldots, x_{k-1}^{**}, x_k^i) \to T^{**}(x_1^{**}, x_2^{**}, \ldots, x_{k-1}^{**}, x_k^{**}),$$

which completes induction.

Now, in the general case, it suffices to define, for $y^* \in F^*$ and $(x_1^{**}, \ldots, x_d^{**}) \in E_1^{**} \times \ldots \times E_d^{**}$,

$$\langle T^{**}(x_1^{**}, \ldots, x_d^{**}), y^*\rangle \stackrel{def}{=} (y^* \circ T)^{**}(x_1^{**}, \ldots, x_d^{**}).$$

It is easily seen that T^{**} is well defined and fulfills all the requirements. □

The assumption in the above theorem is not only sufficient, but also necessary. Let us consider the case $d = 2$ and $F = \mathbb{K}$ for simplicity. For $S \in \mathcal{L}(E_1; E_2^*)$ we have the canonically associated bilinear form given by $T(x_1, x_2) = \langle x_2, S(x_1)\rangle$. The ω^*-separate continuity of T^{**} proves

a).- $T^{**}(x_1^{**}, x_2^{**}) = \langle x_2^{**}, S^{**}(x_1^{**})\rangle$

b).- $S^{**}(x_1^{**})$ is ω^*-continuous, i.e., belongs to E_2^* (identified with a subspace of E_2^{***}).

Thus, $S^{**}(E_1^{**}) \subset E_2^*$, hence S is weakly compact.

The existence of separately weak-star continuous extension to the bidual for *symmetric* multilinear *forms* on a Banach space has been considered in [1], obtaining an analogous result to our Theorem 1 in this case (see [1, Theorem 8.3]). In the same paper, the problem whether, under the same assumptions, *every* continuous multilinear form on E has a separately weak-star continous extension to E^{**}, is posed.

The next result complements Theorem 1:

Corollary 2. *Under the assumptions of Theorem 1, suppose that either*

i) *T is weakly compact,*

or

ii) *for each $i = 1, \ldots d$, every operator from E_i into F is weakly compact.*

Then, for every $T \in \mathcal{L}^d(E_1, \ldots, E_d; F)$, its extension T^{**} takes its values in F.

Proof. The case (i) is clear: if $T(B_{E_1} \times \cdots \times B_{E_d}) = W \subset F$ is weakly relatively compact, by the separate $\omega^* - \omega^*$ continuity of T^{**} and Goldstine's theorem, $T^{**}(B_{E_1^{**}} \times \cdots \times B_{E_d^{**}}) \subset \omega^*$-closure of $W \subset F$.

Suppose now (ii) holds. With the previous notations, $T_d^{**} \stackrel{def}{=} (T_{x_1,\dots,x_{d-1}})^{**}$ and $T^{**}_{x_1,\dots,x_{d-1}}$ are both weak-star continuous from E_d^{**} into F^{**}, and coincide on E_d, hence in all E_d^{**}, i.e.,

$$T^{**}(x_1,\dots,x_{d-1},x_d^{**}) = T_d^{**}(x_d^{**}),\ \forall x_d^{**} \in E_d^{**},$$

and the last term belongs to F by the supposed weak compactness of T_d. Consider now the map $T_{d-1} : E_{d-1} \to F$, given by $T_{d-1}(x) = T^{**}(x_1,\dots,x,x_d^{**})$. Again, is weakly compact and, reasoning as before,

$$T^{**}(x_1,\dots,x_{d-2},x_{d-1}^{**},x_d^{**}) = T_{d-1}^{**}(x_{d-1}^{**}) \in F,\ \forall x_{d-1}^{**} \in E_{d-1}^{**}.$$

The rest is obvious. □

It is worth noticing that the fact that the maps

$$E_i \ni x \mapsto T(x_1,\dots,x_{i-1},x,x_{i+1},\dots,x_d) \in F,$$

for $i = 1,\dots d$ and $x_k \in E_k$ $(k = 1,\dots d)$ fixed, were weakly compact does not suffice to assure that T^{**} is F-valued, as the example $c_o \times c_o \ni (x,y) \mapsto (x_n y_n) \in c_o$ shows. Note also that under (i), T^{**} is weakly compact.

An important particular case in which the assumptions of Theorem 1 are accomplished is when $E_i = C(K_i)$, with K_i a compact Hausdorff space (see [7, ThorŠme 7]). Then Theorem 1 yields a generalization of [10, Theorem 2] and [5, Corollaries 4 and 5] (note that in the last paper, ω^*-convergence of functions always refer to *sequences*). In the following, we shall denote by $\mathcal{B}(K)$ the completion of the simple Borel functions on K under the sup norm (i.e., the bounded Borel functions on K). This space is linearly isometric to a closed subspace of $C(K)^{**}$.

Theorem 3. *Let $K_1,\dots K_d$ be compact Hausdorff spaces, let F be a Banach space and let $T \in \mathcal{L}^d(C(K_1),\dots,C(K_d);F)$. Then there is a unique $\overline{T} \in \mathcal{L}^d(\mathcal{B}(K_1)\dots,\mathcal{B}(K_d);F^{**})$ which extends T and is $\omega^* - \omega^*$ separately continous (the ω^*-topology that we consider in $\mathcal{B}(K_i)$ is the one induced by the ω^*-topology of $C(K_i)^{**}$). Besides, we have*

(1) $\|\overline{T}\| = \|T\|$.

(2) *If $\bar{g}_k = (g_1,\dots,g_{k-1},g_{k+1},\dots g_d)$ with $g_i \in \mathcal{B}(K_i)$, then there is a unique F^{**}-valued bounded ω^*-Radon measure $m_{\bar{g}_k}$ on K_k (i.e., a F^{**}-valued finitely additive bounded vector measure on the Borel subsets of K_k, such that for every $y^* \in F^*$, $(y^* \circ m_{\bar{g}_k})$ is a Radon measure on K_k), verifying*

$$\int g\, d\, m_{\bar{g}_k} = \overline{T}(g_1,\dots,g_{k-1},g,g_{k+1},\dots g_d),\ \forall g \in \mathcal{B}(K_k).$$

(3) *$\overline{T}$ is $\omega^* - \omega^*$ sequentially continuous (i.e., if $(g_i^n)_{n\in\mathbb{N}} \subset \mathcal{B}(K_i)$, $\forall i = 1,\dots,d$, and $g_i^n \xrightarrow{\omega^*} g_i$, then*

$$\lim_{n\to\infty} \overline{T}(g_1^n,\dots,g_d^n) = \overline{T}(g_1,\dots,g_d)$$

in the $\sigma(F^{**},F^*)$ topology.

Proof. Uniqueness is clear, since $\mathcal{B}(K_i)$ is weak-star dense in $C(K_i)^{**}$.

To prove the existence, it suffices to define $\overline{T}$ as the restriction of $T^{**} \in \mathcal{L}^d(C(K_1), \ldots, C(K_d); F^{**})$ to $\mathcal{B}(K_1) \times \ldots \times \mathcal{B}(K_d)$. Then, clearly (1) holds. As for (2), for each $y^* \in F^*$ the map

$$C(K_k) \ni f \mapsto (y^* \circ \overline{T})(g_1, \ldots, g_{k-1}, f, g_{k+1}, \ldots g_d)$$

is a continuous linear form on $C(K_k)$. Hence there exists a unique Radon measure μ_{y^*} on K_k representing it. The formula

$$\langle m_{\bar{g}_k}(A), y^* \rangle = \mu_{y^*}(A), \quad \text{for every Borel subset } A \subset K_k$$

defines then a bounded F^{**}-valued ω^*-Radon measure on K_k and the linear operators from $\mathcal{B}(K_k)$ into F^{**}

$$g \longrightarrow \int g d\, m_{\bar{g}_k}$$

and

$$g \longrightarrow \overline{T}(g_1, \ldots, g_{k-1}, g, g_{k+1}, \ldots g_d)$$

are both weak-star continuous and coincide on $C(K_k)$, hence all over $\mathcal{B}(K_k)$.

Finally, let us prove (3). By composing with $y^* \in F^*$ we can restrict ourselves to the case $F = \mathbb{K}$. We shall use induction on d. For $d = 1$ it is clear. Suppose (3) true for $d-1$. For $g \in \mathcal{B}(K_d)$ and $(f_1, \ldots, f_{d-1}) \in C(K_1) \times \ldots \times C(K_{d-1})$ we define $T_g(f_1, \ldots, f_{d-1}) = \overline{T}(f_1, \ldots, f_{d-1}, g)$. By the uniqueness part of the theorem, it is clear that $\overline{T_g} = (\overline{T})_g$ (obvious notation). Hence, by the induction hypothesis,

$$\lim_{n\to\infty} \overline{T}(g_1^n, \ldots, g_{d-1}^n, g) = \overline{T}(g_1, \ldots, g_{d-1}, g), \ \forall g \in \mathcal{B}(K_d).$$

In particular, the Radon measures (see (2)) $\nu_n = m_{g_1^n, \ldots, g_{d-1}^n}$ converge weakly to $m_{g_1, \ldots, g_{d-1}}$ Then, by [10, Proposition 1(c)],

$$\lim_{n\to\infty} \int g_d^n d\,\nu_n = \int g_d\, dm_{g_1, \ldots, g_{d-1}} = \overline{T}(g_1, \ldots, g_{d-1}, g_d),$$

concluding the proof. □

The above proof of part 3 works whenever the measures $m_{(g_1 \ldots, g_{d-1})}$ considered in part 2 (scalar or vector-valued) are countably additive (see [10, Proposition 1(c)]). By the Orlicz-Pettis theorem, this happens in particular when $\overline{T}$ is F-valued. So we have:

Corollary 4. *With the notations of Theorem 3, if $\overline{T}$ takes its values in F, it is ω^*-norm sequentially continuous.*

From Corollary 2 and well known properties of $C(K)$ spaces we get immediately

Corollary 5. *With the notations of Theorem 3, any of the following conditions implies that $\overline{T}$ takes its values in F:*

i) *T is weakly compact.*
ii) *F contains no copy of c_o.*
iii) *Each K_i is Stonean and F contains no copy of ℓ_∞.*
iv) *Each $C(K_i)$ is a Grothendieck space and F is separable.*

As we mentioned at the introduction, Pelczynski proved in [10], under the assumptions (i) or (ii) above, a particular case of our Theorem 3, obtaining a ω^*-sequentially continuous extension of T to $\mathcal{B}^\Omega(K_1) \times \cdots \times \mathcal{B}^\Omega(K_d)$, where $\mathcal{B}^\Omega(K)$ stands for the bounded *Baire* functions on K. His proof makes use of transfinite induction, starting from the continuous functions, and cannot be extended to cover the case of Borel functions.

This result of Pelczynski was the starting point of the integral representation theorems of Dobrakov. To understand such approach let us begin with the following

Definition 6. [4] *For $i = 1, \dots, d$, let Σ_i be a σ-algebra of subsets of a nonvoid set T_i, and let F be a Banach spaces. A set function $\gamma : \Sigma_1 \times \dots \times \Sigma_d \to F$ is said to be a (countably additive) d-***polymeasure** *if it is finitely (countably) additive in each variable separately. The* **semivariation** *of γ is the set function defined by*

$$\|\gamma\|(A_1, \dots, A_d) = \sup \left\{ \left\| \sum_{j_1=1}^{n_1} \cdots \sum_{j_d=1}^{n_d} a_{1,j_1}, \dots a_{d,j_d} \gamma(A_{1,j_1}, \dots A_{d,j_d}) \right\| \right\} \leq \infty$$

where $(A_{i,j_i})_{j_i=1}^{n_i}, i = 1 \dots d$ is a Σ_i-partition of A_i, and, $\forall i, j_i, |a_{i,j_i}| \leq 1$.

If we denote by $S(\Sigma_i)$ the normed space of the Σ_i-simple functions with the supremum norm and $s_i = \sum_{j_i=1}^{n_i} a_{i,j_i} \chi_{A_{i,j_i}} \in S(\Sigma_i)$, for every F-valued polymeasure γ the formula

$$T_\gamma(s_1, \dots, s_d) = \sum_{j_1=1}^{n_1} \cdots \sum_{j_d=1}^{n_d} a_{1,j_1}, \dots a_{d,j_d} \gamma(A_{1,j_1}, \dots A_{d,j_d})$$

defines a multilinear map $T_\gamma : S(\Sigma_1) \times \dots S(\Sigma_d) \to F$ such that $\|T_\gamma\| = \|\gamma\|(T_1, \dots, T_d)\| \; (\stackrel{def}{=} \|\gamma\|) \; \leq \infty$.

So, if $\|\gamma\| < \infty$, i.e., γ has *finite semivariation*, T_γ can be uniquely extended (with the same norm) to $\mathcal{B}(\Sigma_1) \times \dots \mathcal{B}(\Sigma_d)$, where $\mathcal{B}(\Sigma)$ stands for the completion of $S(\Sigma)$ (i.e., the uniform limits of sequences of Σ simple functions) We will still denote this extension by T_γ) and we shall write also

$$T_\gamma(g_1, \dots, g_d) = \int (g_1, \dots, g_d) d\gamma.$$

It is easily seen that the correspondence $\gamma \mapsto T_\gamma$ is an isometric isomorphism between the space $bpm(\Sigma_1, \dots, \Sigma_d; F)$ of all F-valued polymeasures of finite semivariation, and $L^d(\mathcal{B}(\Sigma_1) \dots \mathcal{B}(\Sigma_d); F)$

This observation, together with Theorem 3, yields immediately the following result:

Theorem 7. *Let $K_1, \dots, K_d$ be compact topological Hausdorff spaces. There exists an isometric isomorphism between $\mathcal{L}^d(C(K_1), \dots, C(K_d); \mathbb{K})$ and the space $M(K_1, \dots, K_d)$ of all countably additive polymeasures γ defined on the product of the Borel σ-algebras of the K_i's (endowed with the semivariation norm), such that, for every $i = 1, \dots, d$ and Borel subsets $A_j \subset K_j$ ($1 \leq j \leq d$) fixed, $\mu_i(A) = \gamma(A_1, \dots, A_{i-1}, A, A_{i+1}, \dots, A_d)$ is a Radon measure on K_i (for the sake of brevity, in the following we shall call this type of polymeasures* Radon polymeasures.*)*

Proof. Given a multilinear map T, it suffices to define $\gamma_T(A_1, \dots, A_d) = \overline{T}(\chi_{A_1}, \dots, \chi_{A_d})$ for $A_i \subset K_i$ Borel.

Conversely, given γ, its associated multilinear form is defined by the formula

$$T_\gamma(f_1, \dots, f_d) = \int (f_1 \dots, f_d) d\,\gamma$$

□

The above theorem extends [5, Corollary, pag. 292], which, being obtained from Pelczynski's result, gives a representation in terms of *Baire* polymeasures. From this representation and our Theorem 7 we immediately get:

Corollary 8. *Every countably additive scalar Baire polymeasure can be uniquely extended to a Radon polymeasure.*

Now we are ready to give our general representation theorem. Let us recall that, from the well known properties of projective tensor products (see, f.i., [6, Chapter VIII]), $\mathcal{L}^d(C(K_1), \dots, C(K_d); \mathbb{K})$ is canonically identified with $(C(K_1)\hat{\otimes} \cdots \hat{\otimes} C(K_d))^*$.

Theorem 9. *Let $K_1, \dots K_d$ be compact Hausdorff spaces, let F be a Banach space, Z a norming subspace of F^* and let $T \in \mathcal{L}^d(C(K_1), \dots, C(K_d); F)$.*

1. *There is a unique $\tilde{T} \in \mathcal{L}^d(\mathcal{B}(K_1) \dots, \mathcal{B}(K_d); Z^*)$ which extends T, such that $\|T\| = \|\tilde{T}\|$, $\tilde{T}$ is $\omega^* - \sigma(Z^*, Z)$ separately continous and $\omega^* - \sigma(Z^*, Z)$ sequentially continuous.*
2. *If we define $\Gamma : \mathcal{B}(K_1) \times \dots, \times \mathcal{B}(K_d) \to Z^*$ by*
$$\Gamma(A_1, \dots, A_d) = \tilde{T}(\chi_{A_1}, \dots \chi_{A_d}),$$
then Γ is a polymeasure of bounded semivariation that verifies:
   (a) $\|T\| = \|\Gamma\|$.
   (b) $T(f_1, \dots, f_d) = \int (f_1, \dots, f_d) d\,\Gamma$ $(f_i \in C(K_i))$
   (c) *For every $z^* \in Z$, $z^* \circ \Gamma \in M(K_1, \dots, K_d)$ and the map $z^* \mapsto z^* \circ \Gamma$ is continuous for the topologies $\sigma(Z, F)$ and $\sigma((C(K_1)\hat{\otimes}, \dots, \hat{\otimes} C(K_d))^*, C(K_1)\hat{\otimes}, \dots, \hat{\otimes} C(K_d))$*

   Conversely, if $\Gamma : \mathcal{B}(K_1) \times \cdots \times \mathcal{B}(K_d) \to Z^*$ is a polymeasure which verifies (c), then it has finite semivariation and formula (b) defines a d-linear continuous operator from $C(K_1) \times \cdots, \times C(K_d)$ into F for which (a) holds.

   Therefore, the correspondence $T \leftrightarrow \Gamma$ is an isometric isomorphism.

Proof. Let us first note that, being Z norming, F is isometrically embedded in Z^*.

Part 1 follows readily from Theorem 3 if we define
$$\langle \tilde{T}(g_1, \dots, g_d), z^* \rangle = \langle \overline{T}(g_1, \dots, g_d), z^* \rangle, \quad (z^* \in Z, \ g_i \in \mathcal{B}(K_i)).$$

Let us pass to part (2). It is clear that Γ is the polymeasure canonically associated to $\tilde{T}$ (see comments previous to Theorem 7); hence, conditions (a) and (b) hold. On the other hand, for $f_i \in C(K_i)$ $(1 \leq i \leq d)$ and $z^* \in Z$,
$$\begin{aligned} \langle T(f_1, \dots, f_d), z^* \rangle &= \langle \int (f_1, \dots, f_d) d\,\Gamma, z^* \rangle \\ = \int (f_1, \dots, f_d) d\,(z^* \circ \Gamma) &= \langle f_1 \otimes \dots \otimes f_d, (z^* \circ T) \rangle, \end{aligned}$$
(identifying $z^* \circ T$ with an element of $(C(K_1)\hat{\otimes} \dots \hat{\otimes} C(K_d))^*$).

Since $T(f_1, \dots, f_d) \in F$, the above expression is, as a function of z^*, $\sigma(Z, F)$-continuous. This means that $z^* \circ \Gamma$ is the representing polymeasure of $z^* \circ T$ and, *via* Theorem 7, that (c) holds.

For the converse, if Γ satisfies (c), the set $\{z^* \circ \Gamma : z^* \in Z, \ \|z^*\| \leq 1\}$ is bounded in $M(K_1, \dots, K_d)$ and hence, from the obvious equality $\|\Gamma\| = \sup\{\|z^* \circ \Gamma\| : \|z^*\| \leq 1\}$

we get that $\|\Gamma\| < \infty$. Let S be the continuous d-linear map associated to Γ. Condition (c) implies that for $f_i \in C(K_i),\ 1 \leq i \leq d$, the map

$$Z \ni z^* \mapsto z^* \circ S(f_1, \ldots, f_d)$$

is $\sigma(Z, F)$-continuous, and so $S(f_1, \ldots, f_d) \in F$. If we define T by formula (b), it takes its values in F and it is clear that $\tilde{T} = S$. In particular, (a) holds. □

Remarks 10.

a).- When F is the dual of a Banach space X, the choice $Z = X$ in the above Theorem contains and extends Theorem 4 of [5], (where only *Baire* polymeasures and weak-star sequentially continuous extensions are considered). In this case, the continuity condition of 2(c) is automatically satisfied for every $F(= X^*)$-valued ω^*-Radon polymeasure Γ (i. e. such thal $x \circ \Gamma \in M(K_1, \ldots, K_d)$ for every $x \in X$). However, this condition cannot be omitted in general, even in the case $d = 1$, as it is well known (for instance, the ℓ_∞-valued, ω^*-Radon polymeasure defined on all the subsets of $\mathbb{N}$ by the formula $m(A) = \chi_A$ is the representing measure of *no* operator $T : \ell_\infty \to c_o$). Hence, Theorem 5 of [5] (which would correspond to our Theorem 9 in the case $Z = F^*$) is not correct as stated (even in the case $d = 1$).

b).- When $Z = F^*$ part 1 of the above theorem is just Theorem 3. Of course we could have proved a similar statement to part 2 of Theorem 3 in the general case. We leave it to the reader.

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